exceed a few MeV. Also the calculation of V in I and II is approximate. In particular, the main contribution to V from the attractive part of $v_{\Lambda N}$, namely that in the P state, has been calculated in an approximation of uncertain accuracy.

Still our results show—at least qualitatively—that there is no serious discrepancy between the calculated and measured binding energy of a Λ-particle in heavy hypernuclei.

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Electron Bremsstrahlung in Scattering by Nuclear Magnetic Moments*†

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The cross section for electron bremsstrahlung in the presence of a magnetic dipole potential is considered, with dependence on photon polarization explicit. Modifications of a result due to Sarkar, to include nuclear spin effects, are derived, and the angular and energy distributions of the radiated quanta are obtained. The related process of pair production is discussed. The infrared divergence is eliminated in the same way as for the Coulomb potential.

I. INTRODUCTION

IN the scattering of electrons by a nucleus, the emission of photons depends on the nuclear magnetic moment as well as on the nuclear charge. Sarkar[1] has obtained the bremsstrahlung cross section corresponding to a spin-independent (i.e., classical) nuclear magnetic moment. It is the purpose of this paper to determine the effects of nuclear spin on the cross section, to obtain the angular and energy distributions of the radiated particles, and to show that, as in the Coulomb case, the infrared divergence is spurious.

The results presented parallel those of Bethe and Heitler,[2] and of Gluckstern, Hull, and Breit[3] for bremsstrahlung in the Coulomb field.

An electromagnetic potential is introduced to represent the nucleus

$$A_\nu(\mathbf{r})=(-\boldsymbol{\mu}\times\boldsymbol{\nabla},ieZ)r^{-1}, \qquad (1.1)$$

where μ and Z are the nuclear magnetic moment and atomic number. The relative magnitude of the magnetic and Coulomb interactions with the electron is considered by Newton,[4] the ratio being

$$\mu|\mathbf{q}|/eZ=(|\mathbf{q}|\mu/\mu_N)/2McZ, \qquad (1.2)$$

where M is the mass of the nucleon, μ_N the nuclear magneton, and $\mathbf{q}$ a momentum transfer characteristic of the scattering process. Evidently, magnetic scattering is of greatest importance for high-energy electrons, the effect decreasing with Z. Unless the momentum transfer is comparable with the nuclear mass, the existence of magnetic properties of the nucleus is almost completely masked by the nuclear charge.

The assumption that the nucleus does not recoil is admittedly unrealistic for very light nuclei, since it is necessary that the experiments be performed at high energies. The most serious violation of this approximation, scattering from the proton, has been considered by Berg and Lindner.[5]

It is interesting to note that polarized targets, suitable for scattering experiments, are currently under investigation.[6]

II. THE DIFFERENTIAL CROSS SECTION

The electromagnetic potential is treated in the first Born approximation. If $(\mathbf{p}_0,iE_0)$ denotes the four-momentum of the incident electron, $(\mathbf{p},iE)$ that of the electron after scattering, then the cross section for emission of a photon with momentum $\mathbf{k}$ and polarization direction $\hat{e}$, is[1]

$$d\sigma=(Z^2e^6/8\pi^2)(kdk/q^4)(p/p_0)\,\mathrm{Tr}(A^++B^+)\times(H+E)(A+B)(H_0+E_0)d\Omega d\Omega_k, \qquad (2.1)$$

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where

$$A=(\pi^2q^2/eZ)\gamma_0\boldsymbol{\gamma}\cdot\hat{e}(H'+E_0)\gamma_0\gamma A(\mathbf{q})/(k\Delta)\,, \tag{2.2}$$

$$B=(\pi^2q^2/eZ)\gamma_0\gamma A(-q)(H''+E)\boldsymbol{\gamma}\cdot\hat{e}\gamma_0/(k\Delta_0)\,, \tag{2.3}$$

$$A_\nu(\mathbf{q})=(2\pi^2q^2)^{-1}(i\boldsymbol{\mu}\times\mathbf{q},\, ieZ)\,. \tag{2.4}$$

The notation is that of reference one, with the exception that the Dirac matrices are anti-Hermitian:

$$\gamma_\mu^+=-\gamma_\mu,\quad \{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu,\nu}\,,\qquad \mu,\nu=1,2,3,4\quad \gamma_4=i\gamma_0\,. \tag{2.5}$$

The momentum transfer to the nucleus is

$$\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k} \tag{2.6}$$

and

$$\Delta=(kE-\mathbf{p}\cdot\mathbf{k})/k,\quad \Delta_0=(kE_0-\mathbf{p}_0\cdot\mathbf{k})/k\,, \tag{2.7}$$

$$p_\mu'=p_\mu+k_\mu,\quad p_\mu''=p_{0\mu}-k_\mu\,. \tag{2.8}$$

That there is no interference between charge and magnetic scattering is evident, after some algebraic manipulation, from Eq. (2.1). Gluckstern *et al.*,[3] and May[7] derived that part of the cross section dependent on the nuclear charge. The cross section for bremsstrahlung in the field of a static (i.e., nonspin-dependent) magnetic dipole is given by Sarkar[1,8]:

$$\begin{aligned}d\sigma=&(e^6/8\pi^2)(M\mu_N)^{-2}(pkdk/p_0)(d\Omega d\Omega_k/q^4)\{2(\mathbf{L}\cdot\hat{e})^2-\tfrac{1}{2}\mathbf{L}^2\{2-\Delta/\Delta_0-\Delta_0/\Delta\\&-(q^2/k^2)[(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\}+4\mathbf{L}\cdot\hat{e}\mathbf{L}\cdot\mathbf{p}_0\mathbf{p}\cdot\hat{e}/k\Delta-4\mathbf{L}\cdot\hat{e}\mathbf{L}\cdot\mathbf{p}\mathbf{p}_0\cdot\hat{e}/k\Delta_0+(\mathbf{L}\cdot\mathbf{p}_0)^2[q^2/\Delta\Delta_0\\&+4(\mathbf{p}\cdot\hat{e})^2/\Delta^2]/2k^2+(\mathbf{L}\cdot\mathbf{p})^2[q^2/\Delta\Delta_0+4(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2]/2k^2-\mathbf{L}\cdot\mathbf{p}\mathbf{L}\cdot\mathbf{p}_0(q^2+4\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e})/k^2\Delta\Delta_0\}\,,\\&\mathbf{L}=\boldsymbol{\mu}\times\mathbf{q}\,.\end{aligned} \tag{2.9}$$

This cross section vanishes if the momentum transfer is parallel to the magnetic moment, that is, if

$$\boldsymbol{\mu}\times\mathbf{q}=0\,. \tag{2.10}$$

III. NUCLEAR SPIN MODIFICATIONS

Newton[4] has pointed out that the static treatment of the nuclear moment is an unnecessary restriction, removed in the following manner. Modification of the interaction to include nuclear spin effects results in replacing $\boldsymbol{\mu}$ in the scattering amplitude by matrix elements of the magnetic moment operator $\boldsymbol{\mu}$ between initial and final nuclear spin states, labeled by Jm and $J'm'$, respectively. The components of the operator $\boldsymbol{\mu}$ do not commute, so that, in the cross section, Eq. (2.9),

$$\boldsymbol{\mu}\cdot\mathbf{a}\boldsymbol{\mu}\cdot\mathbf{b}\rightarrow\tfrac{1}{2}(\boldsymbol{\mu}\cdot\mathbf{a}\boldsymbol{\mu}\cdot\mathbf{b}+\boldsymbol{\mu}\cdot\mathbf{b}\boldsymbol{\mu}\cdot\mathbf{a})\,. \tag{3.1}$$

More explicitly, the replacement is

$$\boldsymbol{\mu}\cdot\mathbf{a}\boldsymbol{\mu}\cdot\mathbf{b}\rightarrow\tfrac{1}{2}\textstyle\sum_{J'm'}(Jm|\boldsymbol{\mu}\cdot\mathbf{a}|J'm')\times(J'm'|\boldsymbol{\mu}\cdot\mathbf{b}|Jm)+(\mathbf{a}\leftrightarrow\mathbf{b})\,. \tag{3.2}$$

The final nuclear spin, not observed, has been summed.

Reduction of the sum on J' and m' is accomplished by using the decomposition and factorization theorems for spherical vector operators,[9] with the result that

$$\textstyle\sum_{J'm'}(Jm|\boldsymbol{\mu}\cdot\mathbf{a}|J'm')(J'm'|\boldsymbol{\mu}\cdot\mathbf{b}|Jm)=\left[\dfrac{(J\|\mathbf{J}\cdot\boldsymbol{\mu}\|J)}{J(J+1)}\right]^2\times\textstyle\sum_{m'}(Jm|\mathbf{J}\cdot\mathbf{a}|Jm')(Jm'|\mathbf{J}\cdot\mathbf{b}|Jm)\,. \tag{3.3}$$

All reference to $\boldsymbol{\mu}$ is contained in the reduced matrix element (the square of) which appears as a multiplicative factor in the cross section.

The nuclear magnetic moment is defined as

$$\mu=(JJ|\boldsymbol{\mu}\cdot\hat{n}|JJ)\,, \tag{3.4}$$

where $\hat{n}$ is the polarization direction in the initial nuclear state (the axis of quantization). This may also be written as

$$\mu=(J\|\mathbf{J}\cdot\boldsymbol{\mu}\|J)/(J+1)\,. \tag{3.5}$$

The sum on m' in Eq. (3.3) is elementary, and equals

$$\mathbf{a}\cdot\mathbf{b}[J(J+1)-m^2]+\mathbf{a}\cdot\hat{n}\mathbf{b}\cdot\hat{n}[3m^2-J(J+1)]\,. \tag{3.6}$$

Combining Eqs. (3.3), (3.5), and (3.6), the replacement to be made in the cross section Eq. (2.9) is

$$\boldsymbol{\mu}\cdot\mathbf{a}\boldsymbol{\mu}\cdot\mathbf{b}\rightarrow\tfrac{1}{2}(\mu/J)^2\{\mathbf{a}\cdot\mathbf{b}[J(J+1)-m^2]+\mathbf{a}\cdot\hat{n}\mathbf{b}\cdot\hat{n}[3m^2-J(J+1)]\}\,. \tag{3.7}$$

In the same way,

$$\boldsymbol{\mu}^2\rightarrow\mu^2(J+1)/J\,. \tag{3.8}$$

Of particular interest is the case J equal to one-half, for no correlation is then predicted with respect to the polarization direction $\hat{n}$.

Since Eq. (3.7) is independent of the sign of m, it is also correct for aligned nuclei.

If the initial state is unpolarized m is averaged. Thus, from Eq. (3.7)

$$\boldsymbol{\mu}\cdot\mathbf{a}\boldsymbol{\mu}\cdot\mathbf{b}\rightarrow\tfrac{1}{3}\mu^2\mathbf{a}\cdot\mathbf{b}(J+1)/J \tag{3.9}$$

and Eq. (3.8) is unchanged.

IV. THE CROSS SECTION FOR POLARIZED NUCLEI

Incorporating Eqs. (3.7) and (3.8) in Eq. (2.9), the cross section for scattering from a polarized nucleus is

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[8] Sarkar's result omits a factor of four.
[9] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), p. 94 ff.

$$
\begin{aligned}
d\sigma = {} & (e^6/8\pi^2)(\mu/M\mu_N)^2(p/p_0)(kdk/q^4)(d\Omega d\Omega_k/2J^2) \\
& \times[\{J(J+1)-m^2\}\{2(\mathbf{q}\times\hat{e})^2+\tfrac{1}{2}q^2\{2-\Delta/\Delta_0-\Delta_0/\Delta-(q^2/k^2)[(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\} \\
& +(4/k)(\mathbf{q}\times\hat{e}\cdot\mathbf{q}\times\mathbf{p}_0\mathbf{p}\cdot\hat{e}/\Delta-\mathbf{q}\times\hat{e}\cdot\mathbf{q}\times\mathbf{p}\mathbf{p}_0\cdot\hat{e}/\Delta_0)+q^2(\mathbf{q}\times\mathbf{k})^2/2k^2\Delta\Delta_0+(2/k^2)[(\mathbf{q}\times\mathbf{p}_0)^2(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{q}\times\mathbf{p})^2 \\
& \times(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{q}\times\mathbf{p}\cdot\mathbf{q}\times\mathbf{p}_0\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\}+\{3m^2-J(J+1)\}[2(\mathbf{q}\times\hat{e}\cdot\hat{n})^2+\tfrac{1}{2}(\mathbf{q}\cdot\hat{n})^2\{2-\Delta/\Delta_0-\Delta_0/\Delta-(q^2/k^2) \\
& \times[(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\}+(4\mathbf{q}\times\hat{e}\cdot\hat{n}/k)(\mathbf{q}\times\mathbf{p}_0\cdot\hat{n}\mathbf{p}\cdot\hat{e}/\Delta-\mathbf{q}\times\mathbf{p}\cdot\hat{n}\mathbf{p}_0\cdot\hat{e}/\Delta_0) \\
& +q^2(\mathbf{q}\times\mathbf{k}\cdot\hat{n})^2/2k^2\Delta\Delta_0+(2/k^2)[(\mathbf{q}\times\mathbf{p}\cdot\hat{n})^2(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2+(\mathbf{q}\times\mathbf{p}_0\cdot\hat{n})^2(\mathbf{p}\cdot\hat{e})^2/\Delta^2-2\mathbf{q}\times\mathbf{p}\cdot\hat{n}\mathbf{q}\times\mathbf{p}_0\cdot\hat{n}\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]] \\
& -J(J+1)q^2\{2-\Delta/\Delta_0-\Delta_0/\Delta-(q^2/k^2)[(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\}]. \qquad (4.1)
\end{aligned}
$$

This result, for the emission of a photon in the presence of a polarized nucleus of spin J, complements that derived by May[7] for the Coulomb potential.

The quantum-mechanical condition corresponding to Eq. (2.10) is

$$\hat{n}\times\mathbf{q}=0. \qquad (4.2)$$

When Eq. (4.2) is valid, the cross section Eq. (4.1) has the form

$$
\begin{aligned}
d\sigma = {} & (e^6/16\pi^2)(\mu/\mu_N M)^2(p/p_0)(kdk/q^2)d\Omega d\Omega_k\{[J(J+1)-m^2]/J^2\}[-2(\hat{n}\cdot\hat{e})^2+2\{\Delta/\Delta_0+\Delta_0/\Delta \\
& -(q^2/k^2)[(\mathbf{p}\cdot\hat{e})^2/\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]\}+(4/k)(\hat{n}\times\hat{e}\cdot\hat{n}\times\mathbf{p}_0\mathbf{p}\cdot\hat{e}/\Delta-\hat{n}\times\hat{e}\cdot\hat{n}\times\mathbf{p}\mathbf{p}_0\cdot\hat{e}/\Delta_0) \\
& +(\mathbf{q}\times\hat{k})^2/2\Delta\Delta_0+(2/k^2)[(\hat{n}\times\mathbf{p}_0)^2(\mathbf{p}\cdot\hat{e})^2/\Delta_0^2+(\hat{n}\times\mathbf{p})^2(\mathbf{p}_0\cdot\hat{e})^2/\Delta_0^2-2\hat{n}\times\mathbf{p}\cdot\hat{n}\times\mathbf{p}_0\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}/\Delta\Delta_0]]. \qquad (4.3)
\end{aligned}
$$

The cross section vanishes identically in the classical limit, $J\rightarrow\infty$, in agreement with the result of Sec. II. This, of course, is a manifestation of the correspondence principle.

In general, the cross section Eq. (4.3) is not identically zero, even if the nucleus is completely polarized. The quantity

$$(Jm|\mathbf{J}^2-(\mathbf{J}\cdot\hat{n})^2|Jm)=J(J+1)-m^2$$

occurring in Eq. (4.3) (this is a measure of the deviation of the component of angular momentum perpendicular to $\hat{n}$, from the expectation value zero) is positive, even if $|m|$ is equal to J.

V. THE CROSS SECTION FOR UNPOLARIZED NUCLEI

Summing Eq. (4.1) over m, or, equivalently, introducing the replacements Eqs. (3.8) and (3.9) in Eq. (2.9), gives the cross section for scattering from an unpolarized nuclear target.

$$
\begin{aligned}
d\sigma = {} & (e^6/8\pi^2)[(J+1)/3J](\mu/M\mu_N)^2(p/p_0)(dk/k)(d\Omega d\Omega_k/q^2)[\mathbf{p}\cdot\hat{e}\mathbf{p}_0\cdot\hat{e}(4m^2-4EE_0-q^2)/\Delta\Delta_0 \\
& +(\mathbf{p}\cdot\hat{e})^2(q^2+4p_0^2)/2\Delta^2+(\mathbf{p}_0\cdot\hat{e})^2(q^2+4p^2)/2\Delta_0^2+\tfrac{1}{2}k^2(\Delta/\Delta_0+\Delta_0/\Delta+2+q^2/\Delta\Delta_0)]. \qquad (5.1)
\end{aligned}
$$

In the limit $J\rightarrow\infty$, Eq. (5.1) becomes the classical result.[1] A similar remark pertains to Eq. (4.1), for a completely polarized nucleus, $|m|=J$.

VI. PHOTON POLARIZATION SUM

Summation over the polarization directions of the photon, in Eqs. (4.1) and (5.1), leads to magnetic analog of the Bethe-Heitler formula. The cross section for scattering by a classical magnetic moment distribution is, from Eq. (2.9)

$$
\begin{aligned}
d\sigma = {} & (e^6/8\pi^2)(M\mu_N)^{-2}(p/p_0)(kdk/q^4)d\Omega d\Omega_k\{2(\mathbf{L}\times\hat{k})^2-(\mathbf{L}^2)(2-\Delta/\Delta_0-\Delta_0/\Delta) \\
& +(q^2/2k^2)(\mathbf{L}^2)[(\mathbf{p}\times\hat{k})^2/\Delta^2+(\mathbf{p}_0\times\hat{k})^2/\Delta_0^2-2\mathbf{p}_0\times\hat{k}\cdot\mathbf{p}\times\hat{k}/\Delta\Delta_0]+4(\mathbf{L}\cdot\mathbf{p}_0/k\Delta)(\mathbf{L}\cdot\mathbf{p}-\mathbf{L}\cdot\hat{k}\mathbf{p}\cdot\hat{k}) \\
& -4(\mathbf{L}\cdot\mathbf{p}/k\Delta_0)(\mathbf{L}\cdot\mathbf{p}_0-\mathbf{L}\cdot\hat{k}\mathbf{p}_0\cdot\hat{k})+k^{-2}(\mathbf{L}\cdot\mathbf{p}_0)^2[q^2/\Delta\Delta_0+2(\mathbf{p}\times\hat{k})^2/\Delta^2]+k^{-2}(\mathbf{L}\cdot\mathbf{p})^2 \\
& \times[q^2/\Delta\Delta_0+2(\mathbf{p}_0\times\hat{k})^2/\Delta_0^2]-2\mathbf{L}\cdot\mathbf{p}\mathbf{L}\cdot\mathbf{p}_0(q^2+2\mathbf{p}\times\hat{k}\cdot\mathbf{p}_0\times\hat{k})/k^2\Delta\Delta_0\}. \qquad (6.1)
\end{aligned}
$$

If the nucleus is unoriented, the cross section, from Eq. (5.1), is

$$
\begin{aligned}
d\sigma = {} & (e^6/8\pi^2)[(J+1)/3J](\mu/M\mu_N)^2(p/p_0)(dk/k)(d\Omega d\Omega_k/q^2)[\mathbf{p}_0\times\hat{k}\cdot\mathbf{p}\times\hat{k}(4m^2-4EE_0-q^2)/\Delta\Delta_0 \\
& +(\mathbf{p}\times\hat{k})^2(q^2+4p_0^2)/2\Delta^2+(\mathbf{p}_0\times\hat{k})^2(q^2+4p^2)/2\Delta_0^2+k^2(\Delta/\Delta_0+\Delta_0/\Delta+2+q^2/\Delta\Delta_0)]. \qquad (6.2)
\end{aligned}
$$

VII. THE INTEGRATED BREMSSTRAHLUNG CROSS SECTION

In this section, the bremsstrahlung cross section for scattering from an unpolarized nucleus, Eq. (5.1), is integrated over the direction of the scattered electron. The integrals involved are discussed in the appendix to Ref. 10.

The notation of Gluckstern and Hull[10] is adopted:

$$\epsilon=\ln[(E+p)/(E-p)],\quad \epsilon_0=\ln[(E_0+p_0)/(E_0-p_0)],\quad \epsilon^T=\ln[(T+p)/(T-p)],$$
$$\mathbf{T}=\mathbf{p}_0-\mathbf{k},\quad L=\ln[(EE_0-m^2+pp_0)/(EE_0-m^2-pp_0)]. \tag{7.1}$$

Integration of Eq. (5.1), after considerable simplification, yields the following cross section:

$$d\sigma=(e^6/4\pi)[(J+1)/3J](\mu/M\mu_N)^2(p/p_0)(dk/k)d\Omega_k$$
$$\times\{(\epsilon^T/pT)[-2m^2T^2/\Delta_0^2+(k^2/T^2)(k\Delta_0+p^2+kE)]+(\epsilon/p\Delta_0)(E\Delta_0+p_0^2+p^2)$$
$$+(L/2\Delta_0^2pp_0)[k^2\Delta_0^2+2m^2k\Delta_0+2m^2(EE_0-m^2)]+m^2/\Delta_0^2-3-(\Delta_0-E)(k^2/\Delta_0T^2)\}. \tag{7.2}$$

The result, Eq. (7.2), assumes that the polarization of the photon is not observed. If this polarization is observed, then it is sufficient to consider only photons polarized in the $\mathbf{p}_0\mathbf{k}$ plane,[10] that is,

$$\hat{e}=[kt^2\mathbf{a}+E(T^2-p^2-2kE)\mathbf{b}]/2|\mathbf{p}_0\times\mathbf{k}|, \tag{7.3}$$

$$\mathbf{a}=2\mathbf{T}/t^2,\quad \mathbf{b}=\hat{k}/E,\quad t^2=T^2+p^2. \tag{7.4}$$

The cross section, for the production of photons polarized according to Eq. (7.3), is

$$d\sigma=(e^6/4\pi)(\mu/M\mu_N)^2(p/p_0)(dk/k)d\Omega_k\{(\epsilon^T/pT)[-4m^2k/\Delta_0-2m^2p^2/\Delta_0^2+Ek+3EE_0-2m^2+2(E-\Delta_0)^2/\sin^2\vartheta_0$$
$$+(k^2/2T^2)(k\Delta_0+p^2+Ek)]+(\epsilon/p)\{(2EE_0+k^2-2m^2)/\Delta_0+\tfrac{1}{2}[-E\Delta_0^2+2\Delta_0(EE_0-p_0^2)$$
$$+E(2p_0^2-m^2)]/(\mathbf{p}_0\cdot\hat{e})^2\}+(L/2\Delta_0pp_0)[k^2\Delta_0/2+\Delta_0^{-1}(\mathbf{p}_0\cdot\hat{e})^{-2}(E_0\Delta_0-m^2)$$
$$\times[-2E_0k\Delta_0^2+2\Delta_0(2m^2k-Ep_0^2)+2m^2(EE_0-m^2)]]+k^2(E-\Delta_0)/2\Delta_0T^2-2\}. \tag{7.5}$$

These two results, Eqs. (7.2) and (7.5), correspond to $d\sigma_{\rm I}$ and $d\sigma_{\rm II}$ (for the Coulomb potential) in the work of Gluckstern and Hull.

Finally, the cross section, Eq. (7.2), is integrated over the direction of the photon. From this integration results the energy spectrum of the emitted radiation, which, again after much simplification, is

$$d\sigma=e^6[(J+1)/6J](\mu/M\mu_N)^2(p/p_0)(dk/k)$$
$$\times[(kL/pp_0)(2k+m^2\epsilon_0/p_0-m^2\epsilon/p)$$
$$+(\epsilon\epsilon_0/pp_0)(p^2+p_0^2)]. \tag{7.6}$$

The symmetry of this equation under interchange of $(\mathbf{p}_0,iE_0)$ and $(\mathbf{p},iE)$ is to be expected from detailed balance, together with the fact that integration over the direction of both final particles is symmetric under the same interchange $(k=E_0-E\to -k)$.

The cross section, Eq. (6.1), for polarized nuclei, has also been integrated over the direction of the scattered electron, but the result, which is rather long, will not be given here.

VIII. POLARIZATION DEPENDENCE FOR FIXED ELECTRON RECOIL DIRECTION

Gluckstern *et al.*[3] show that, with the direction of the scattered electron fixed, the dependence of the bremsstrahlung cross section, for the Coulomb potential, on the polarization direction of the photon is

$$d\sigma=[A+B\cos2\phi+C\sin2\phi]d\Omega d\Omega_k$$
$$=[A-D+2D\cos^2(\phi-\phi_0)]d\Omega d\Omega_k, \tag{8.1}$$

where $D=(B^2+C^2)^{1/2}$, $\tan2\phi_0=C/B$. ϕ is the direction of $\hat{e}$ in a plane perpendicular to $\mathbf{k}$. Reference to Eq. (4.1) shows that a similar relation is valid for the magnetic process. Thus, the cross section including both effects has the form of Eq. (8.1), but, of course, with different coefficients, A, B, C, D.

The analysis of Ref. 3 demonstrates that the radiation can be interpreted as consisting of an unpolarized part and a linearly polarized part, with intensities proportional to $A-D$ and $2D$, respectively.

IX. PAIR PRODUCTION

The bremsstrahlung and pair production cross sections are connected by the (four-momentum) substitutions[11]:

$$p_0\to p_-,\quad p\to -p_+,\quad k\to k. \tag{9.1}$$

Combining Eqs. (2.1) and (9.1), and making the requisite change in the density of states factor, the cross section for production of an electron-positron pair of momenta $\mathbf{p}_-$ and $\mathbf{p}_+$, from a photon of momentum $\mathbf{k}$, is

$$d\sigma=-(e^6/4\pi^2)(M\mu_N)^{-2}(p_+p_-dE_+/k)(d\Omega_+d\Omega_-/q^4)$$
$$\times\{2(\mathbf{L}\cdot\hat{e})^2-\tfrac{1}{2}(\mathbf{L})^2\{2+\Delta_+/\Delta_-+\Delta_-/\Delta_+-(q^2/k^2)[(\mathbf{p}_+\cdot\hat{e})^2/\Delta_+^2+(\mathbf{p}_-\cdot\hat{e})^2/\Delta_-^2-2\mathbf{p}_+\cdot\hat{e}\mathbf{p}_-\cdot\hat{e}/\Delta_+\Delta_-]\}$$
$$+4\mathbf{L}\cdot\hat{e}\mathbf{L}\cdot\mathbf{p}_-\mathbf{p}_+\cdot\hat{e}/k\Delta_++4\mathbf{L}\cdot\hat{e}\mathbf{L}\cdot\mathbf{p}_+\mathbf{p}_-\cdot\hat{e}/k\Delta_-+2(\mathbf{L}\cdot\mathbf{p}_-)^2(\mathbf{p}_+\cdot\hat{e})^2/k^2\Delta_+^2$$
$$+2(\mathbf{L}\cdot\mathbf{p}_+)^2(\mathbf{p}_-\cdot\hat{e})^2/k^2\Delta_-^2+4\mathbf{L}\cdot\mathbf{p}_+\mathbf{L}\cdot\mathbf{p}_-/k^2\Delta_+\Delta_--q^2(\mathbf{L}\cdot\hat{k})^2/2\Delta_+\Delta_-\}. \tag{9.2}$$

[10] R. L. Gluckstern and M. H. Hull, Phys. Rev. **90**, 1030 (1953).

[11] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955).

In Eq. (9.2):

$$\mathbf{q}=\mathbf{p}_-+\mathbf{p}_+-\mathbf{k}\,,$$

$$\Delta_+=(kE_+-\mathbf{p}_+\cdot\mathbf{k})/k\,,\quad \Delta_-=(kE_--\mathbf{p}_-\cdot\mathbf{k})/k\,. \tag{9.3}$$

All of the results presented in the preceding sections can be modified in the same way except Eq. (7.6), which involves integration over the photon direction.

X. THE INFRARED DIVERGENCE

The bremsstrahlung cross section is characterized by a factor dk/k, where k is the photon energy. Integration with respect to this variable ($0\leq k\leq E_0-m$) leads to a divergence in the limit $k\rightarrow 0$. That this divergence is actually nonexistent in an experimentally observable cross section was shown first by Schwinger[12] for the case of single photon production in the Coulomb field. The argument is that an experiment cannot distinguish between a true elastic scattering event and a process in which an extremely soft photon (with energy less than some minimum $\Delta\epsilon$, determined by the resolution of the experimental apparatus) is emitted. Thus, the elastic cross section and the cross section for soft photon emission must be combined in order to calculate a cross section to be compared with experiment. Since the bremsstrahlung cross section is of order α^2A^2, where A_μ is the vector potential, it is also necessary to include the radiative correction to elastic scattering (since the cross term between this amplitude and the elastic amplitude is of order α^2A^2), as well as the effect of vacuum polarization.

A. The Mass Operator

Calculation of the radiative correction to elastic scattering is based on the concept of the mass operator. The formulation considered[13] depends on a modification of the Dirac equation of the following form:

$$(\gamma\pi+M)\psi=0\,, \tag{10.1}$$

$$\pi_\mu=p_\mu-eA_\mu\,. \tag{10.2}$$

The effect of the external field A_μ on the motion of the electron, with all virtual processes excluded, is determined by π_μ. M, the mass operator, symbolizes all (virtual) radiative effects in the field A_μ. In position representation, the mass operator, correct to first order in e^2 (and exact in the external field A_μ), is[14]

$$M(x,x')=m\delta(x-x')+ie^2\gamma_\mu G(x,x')\gamma_\mu D(x,x')\,. \tag{10.3}$$

G is the (out-going wave) Green's function for the electron, and D the photon propagator.

Newton[13] has considered the operator form of this equation

$$M=m+ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ikx}\gamma_\mu(m+\gamma\pi)^{-1}\gamma_\mu e^{-ikx}\,, \tag{10.4}$$

and developed an expansion in powers of eA. The one-photon mass operator, Eq. (10.3), to first order in eA, is

$$M_1=m+\frac{i\alpha}{4\pi}\int_0^\infty ds\int_0^1 du e^{-isum^2}\times e\int\frac{d^4k}{(2\pi)^2}e^{ikx}m_1(u,s,k)\,, \tag{10.5}$$

where

$$m_1(u,s,k)=\int_0^1 dv e^{-isuv(1-v)k^2}\times\{mu(u-1)\sigma F+[2-u+2u^2v(1-v)-4ism^2u(u^2-1)v(1-2v)]\gamma J\}\,. \tag{10.6}$$

$F_{\mu\nu}$ and J_μ are Fourier transforms of the field and current operators, defined by

$$A_\mu\equiv A_\mu(k)=\int\frac{d^4x}{(2\pi)^2}A_\mu(x)e^{-ikx}. \tag{10.7}$$

In both Eq. (10.5) and Eq. (10.7), x is an operator.

B. The Cross Section

Newton[13] gives a derivation of the differential scattering cross section in a context facilitating calculations with the mass operator. A Dirac equation of the form

$$(\gamma p+m+\mathcal{H})\psi=0 \tag{10.8}$$

is considered, with $\mathcal{H}$ describing all effects of the electromagnetic field. If $\mathbf{p}$ and $\mathbf{q}$ are the momenta of the electron before and after scattering, respectively, than the cross section is

$$d\sigma/d\Omega=2\pi^4\,\mathrm{Tr}(m-\gamma p)(\mathbf{p}|H|\mathbf{q})\times(m-\gamma q)(\mathbf{q}|\gamma_0H^+\gamma_0|\mathbf{p})\,, \tag{10.9}$$

$$H=(1+\mathcal{H}G_0)^{-1},\quad G_0=(m+\gamma p)^{-1}. \tag{10.10}$$

The interaction

$$\mathcal{H}=-e\gamma A-e\gamma A'+M_1 \tag{10.11}$$

accounts for both elastic scattering and a radiative process involving one virtual photon, in addition to the vacuum polarization term, A_μ', related to A_μ according to[15]

$$A_\mu'(k)=-\frac{\alpha}{2\pi}A_\mu(k)\int_0^1 dv(1-v^2)\times\int_0^\infty\frac{ds}{s}\exp\left\{-is\left[m^2+\frac{k^2}{4}(1-v^2)\right]\right\}\,. \tag{10.12}$$

[12] J. Schwinger, Phys. Rev. 76, 790 ff. (1949), especially p. 812.
[13] R. G. Newton, Phys. Rev. 94, 1773 (1954).
[14] J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452 (1951).
[15] J. Schwinger, Phys. Rev. 82, 678 (1951).

Matrix elements of the mass operator are calculated from Eq. (10.5):

$$(\mathbf{p}|M_1|\mathbf{q})=a_1\gamma J(\mathbf{k})+ia_2\sigma F(\mathbf{k}), \tag{10.13}$$

$$\mathbf{k}=\mathbf{p}-\mathbf{q}. \tag{10.14}$$

The (field-independent) coefficients, a_1 and a_2, are also determined from Eq. (10.5).

Because of a nonallowed expansion in powers of eA, the mass operator is actually infrared divergent. Divergences of this type are discussed in detail by Schwinger[12] and by Newton,[13,16] who show that introduction of a nonzero photon mass ϵ makes the expansion valid and, at the same time, prevents the(integrated) bremsstrahlung cross section from diverging. The physical reason for the divergences is that the photon mass is zero, and the procedure for combining the divergent amplitudes for radiative processes is to assume a photon mass ϵ and to attempt to eliminate terms diverging as $\epsilon\rightarrow 0$.

The result of the calculation, which is straightforward, is that the cross section for scattering in an arbitrary electromagnetic potential A_μ, including radiative effects to first order, is

$$\sigma_{\text{exp}}(\theta)=[1-\delta(\theta,\Delta\epsilon)]\sigma(\theta). \tag{10.15}$$

Since $\delta(\theta,\Delta\epsilon)$ is independent of the potential, it is identical with the function derived by Schwinger[12] and Newton[13] for Coulomb scattering.

[16] R. G. Newton, Phys. Rev. **96**, 1523 (1954).

As is easily verified from Eq. (10.9), the elastic cross section for scattering by a magnetic dipole distribution is

$$\sigma(\theta)=4\pi^2e^4(M\mu_N)^{-2}(\mathbf{p}-\mathbf{q})^{-4}\times\{(\mathbf{p}-\mathbf{q})^2[\boldsymbol{\mu}\times(\mathbf{p}-\mathbf{q})]^2+4(\boldsymbol{\mu}\times\mathbf{p}\cdot\mathbf{q})^2\}. \tag{10.16}$$

The modifications for nuclear spin (Sec. III) apply also to this result.

The possibility of the emission of very soft quanta (which need not be observed, and, in fact, are assumed unobservable) thus necessitates a correction to the elastic cross section. This correction simultaneously permits elimination of the infrared divergence associated with bremsstrahlung.

Since Eq. (10.15) already includes emission of quanta with energy less than $\Delta\epsilon(\Delta\epsilon\ll m)$, the cross section for electron scattering with energy loss not greater than $\Delta E(\Delta\epsilon\leq\Delta E\leq E_0-m)$ is the sum of Eqs. (10.15) and (4.1), the latter having been integrated over the photon variables with $\Delta\epsilon\leq k\leq\Delta E$. The dependence on $\Delta\epsilon$ then cancels and the cross section is finite. If $\Delta E=E_0-m$, the cross section with the final electron energy not observed is obtained. The resulting integrals, however, are rather complicated.

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